ELECTRONICS FOR ENGINEERS
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PREAMPLIFIER

For semiconductor detectors or ionisation chambers, the charge deposited in the detector

$Q = (q * E) / \varepsilon$ where E is the energy deposited in the detector and ε is the average energy required to produce an ion-pair.

q = $1.6 * 10^{-19}$ Coulomb electron charge

E ~ MeV for γ-rays and 10- 100 MeV for energetic ions

ε ~ 3 eV for semiconductors, and 30 eV for gas ionisation counters.

input signal : $V = Q/C$ where Q is charge collected by the detector and C is the total capacitance (detector + preamp).

Since the capacitance is typically 10 - 100 pF, the signal level is ~ mV only. To minimise the capacitance associated with a long cable (101 pF/m for RG58 and 44 pF/m for RG62 cable) the preamp should be located close to the detector.

A preamplifier has to preserve

- maximum signal to noise ratio
- minimum change in shaping
- information content in the signal (i.e. energy & timing)

Preamplifiers are mainly of two types (i) voltage sensitive & (ii) current sensitive. The first is used for timing applications or for photomultiplier signals. The second is used when the detector capacitance may vary with time degrading output resolution.

The rise time of the voltage pulse is affected by (i) charge collection time in the detector and (ii) rise time of the amplification section. The first may vary from ~ ns for small detectors to ~ 500 ns for large coaxial Ge detectors. The collection time depends on the size of the detector, electric filed and electron/hole mobility.

For a detector in which an average charge of Q is deposited per event, a count rate of N particles/sec constitutes an average current of I = QN. In the absence of any restorative mechanism, the gate voltage will rise towards the HV. In charge sensitive preamp with resistive feedback, a resistance R_f is placed in parallel with the feedback capacitance C_f to allow the output signal to decay with a time constant $\tau = R_f C_f$. Typical values of τ is ~ 50 μsec. This resistance however contributes to noise and for extremely low noise application transistor feedback or opto-feedback circuits are employed. The output signal of a charge sensitive preamp is $V_{out} = -Q/C_f$ where C_f is the feedback capacitance.

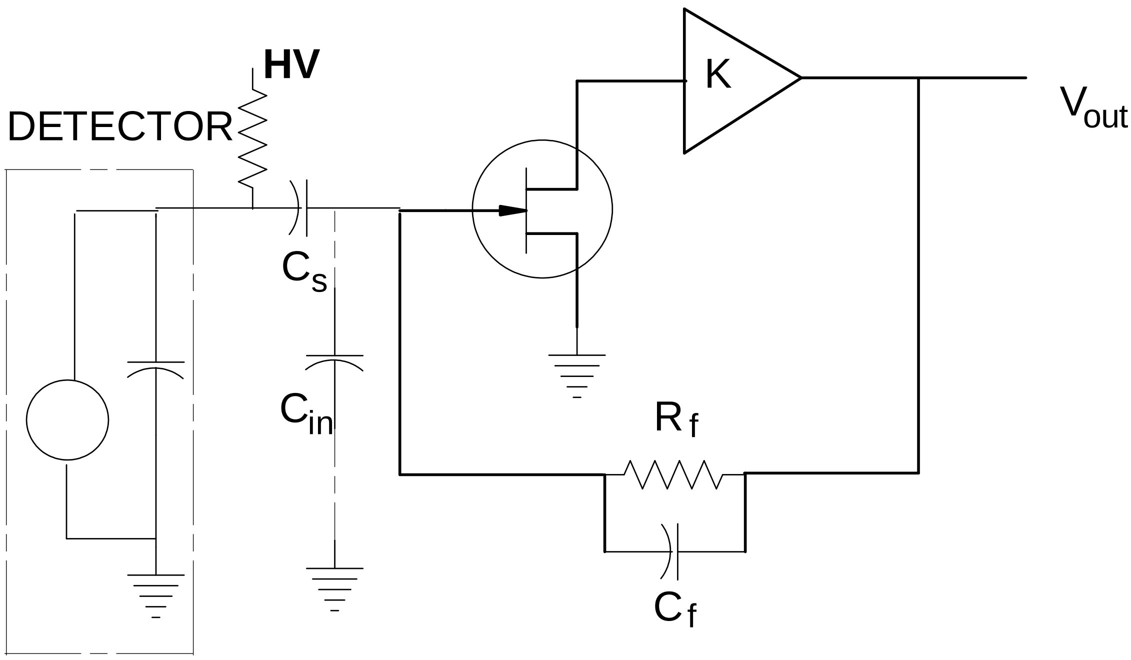


Fig 1: CHARGE SENSITIVE PREAMP

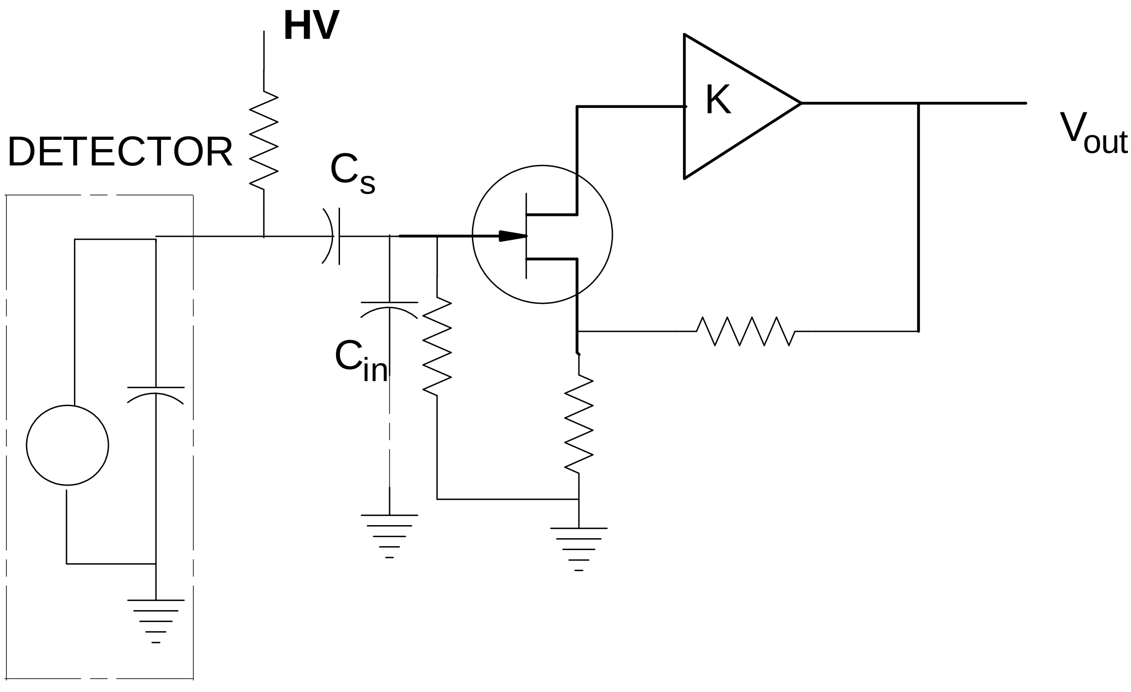


Fig 2: VOLTAGE SENSITIVE PREAMP

The gain stage of the preamplifier must have large gain in order provide a large dynamic capacitance to the input stage. For a gain K, the effective input capacitance with feedback is KC_f which must be large compared to the detector capacitance. In order to provide a fast rise time of the output pulse, the gain-bandwidth product of the amplifier should also be large. The rise time (10% - 90%) of the output pulse is :

$$\left[t_r = 2.2 \frac{(C_{in} + C_D)}{K \omega_h C_f} \right]$$

where ω_h is the angular frequency where the gain falls by 3 db. For detectors with large input capacitance, the rise time is adversely affected.

A typical design of the preamplifier is shown in fig 3. To reduce the Miller effect in gate-drain capacitance, a cascode connection of a common base transistor configuration is used. The constant current source at the emitter side may be replaced by a resistance or large inductance. The buffer with high input and low output impedance is made up of a npn-pnp transistor emitter follower combination. To reduce cross-over distortion, an identical pair of npn-pnp transistors are configured as diodes to bias the two halves of the emitter follower [see, for example, the circuit of PA422 in *R. Bassini et al, Nucl. Instr. Meth. A305(1991)449*] . To protect the gate of the input FET from breakdown if the gate voltage swing is too high, a reverse biased FET configured as a diode is connected to the gate.

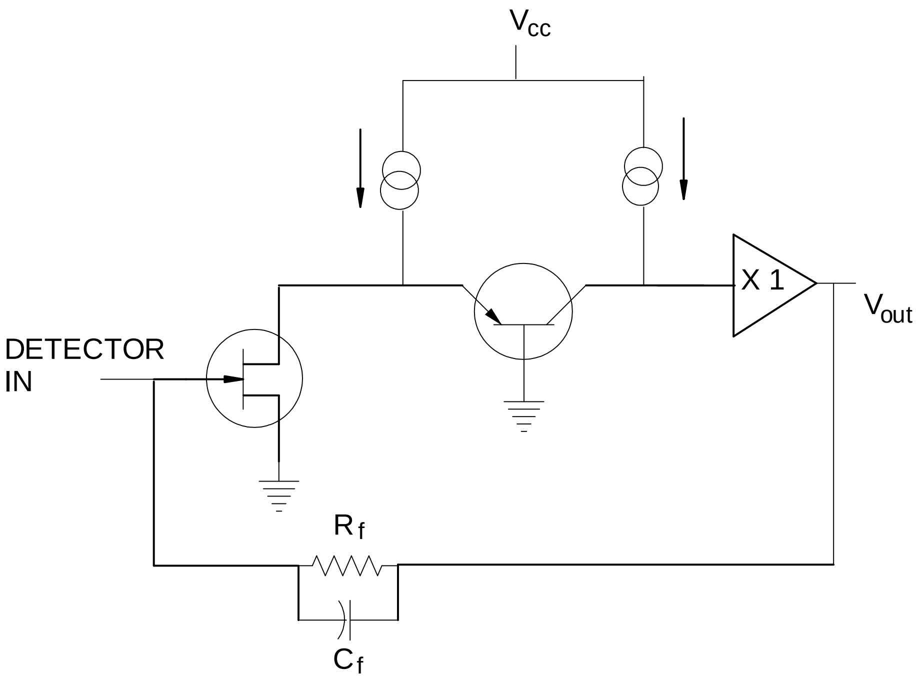


Fig 3: Charge sensitive preamp

NOISE IN PREAMPS

The main sources of noise in a preamplifier comes from its input where the signal levels are at their lowest. Three different sources of noise exist (i) thermal (ii) shot and (iii) flicker noise.

Parallel noise (Step Noise)

Let us first consider the noise sources that are shunted by the input capacitance of the preamp + detector combination (fig 4). Thermal noise is associated with any resistive element and is the result of the random thermal motion of electrons in the conducting material. The r.m.s. value of the noise is:

$\delta V_p^2 = 4kTR\Delta f$ where k is the Boltzmann constant (= 8.62 . 10^{-5} eV/K) and Δf is the frequency bandwidth. Normally the resistance is in parallel to the signal source (R_f and R_{HV} in fig 1) so that the corresponding current noise is:

$\delta I_p^2 = 4kT\Delta f / R$. As a result, both the feedback resistor R_f and the biasing resistor R_{HV} should be high for low noise applications. This noise source acts in parallel to the input capacitance ($C_d + C_{in} + C_f$) of the detector + preamp combination.

In a semiconductor detector at room temperature, the leakage current is also a source of noise:

$\delta I_p^2 = 2q\, I_d\, \Delta f$ where I_d is the leakage current of the detector. The total parallel noise current is:

$$\boldsymbol{\delta I_p^2 = (2q\, I_d + 4kT/R_{eq})\, \Delta f}$$

where R_{eq} is the parallel combination of all shunting resistances.

The parallel noise is also known as **step noise** as due to the integrating effect of the capacitance, it appears as a series of steps in a staircase.

Series Noise (Delta Noise)

Unlike the step noise which is generated mainly external to the preamplifier, the FET itself contributes to a white noise. This noise comes in series with the detector signal and is also known as **delta noise** as it corresponds to bursts of pulses of very short duration.

Shot noise exists in a BJT (Transistor) because the emission of electrons and holes across a pn junction is random in nature. It is expressed by :

$\delta I_s^2 = 2qI_e\, \Delta f$ where q is the electronic charge and I_e is the collector current. To obtain an equivalent input voltage source V_s , one can calculate the effect of the base voltage on the collector current I_c :

$\delta V_s = I_c/g_m$ where g_m is the transconductance of the transistor.

This gives rise to a shot noise $\delta V_n^2 = 4kT\Delta f\ (r_x + 1/2\ g_m)$ for a preamplifier with BJT front end where r_x is the base resistance of the transistor.

For a FET, there is no shot noise, but the resistive channel material gives rise to a white thermal noise source:

$$\delta V_s^2 = 4kT\Delta f\ (0.7/g_m)\ .$$

The contributions from the different noise sources depend on whether they are in series or in parallel with the signal. The biasing resistance, feedback resistance and leakage current contributions are in parallel to the signal while the contribution from the input BJT/FET is in series. The input signal and the parallel noise are shunted by the input capacitance of the detector / FET combination.

The input detector signal and the parallel noise sources are shunted by the input admittancc $Y_D = j\omega\ C_{tiotal}$ of the detector & FET combination. The series noise (or delta noise) contribution can therefore be written as an input current noise source:

$$\boldsymbol{\delta I_\Delta^2 = 4\gamma kT\ (\omega\ C_{total})^2\ \ \Delta f\ /g_m}$$

where γ is 0.5 or 0.7 depending on whether it is BJT or FET. The extra ω^2 dependence in the delta noise contribution implies that its effect is more important at high frequencies, corresponding to short sampling times.

In addition to the above two main sources of noise, most electronic devices produce a flick noise at low frequency $\propto \Delta f/f$ which is of importance only at very low frequencies. This is more important for MOSFETs as compared to JFETs. In addition, there is noise contribution due to recombination of traps present in the gate region. For Si-based FETs, the recombination noise contribution is minimum at around 100 - 140° Kelvin. The total noise contribution is obtained by adding the individual noise contributions in quadrature.

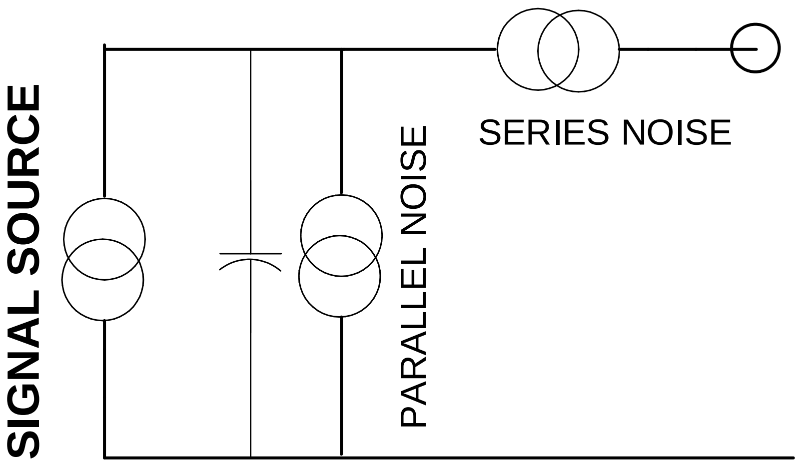


Fig 4 : Noise sources in a preamplifier

The noise contributions in the above case have been calculated in the absence of any feedback. It can be shown that in the presence of a further noise-less amplification stage, both signal and the noise are equally affected. Consequently the signal to noise ratio remain unaffected by feedback. It is therefore desirable to always express the noise in terms of an effective input signal. The equivalent noise charge (ENC) is obtained by integrating the noise current over the time domain. Since the noise term depends explicitly depends on the band width Δf, the signal to noise ratio can be improved by optimising the processing time.

Fig 5 shows the complete signal processing chain for a passive time-invariant pulse shaper. Both the signal and the noise are processed by the same shaper in order to achieve the best signal to noise ratio. The individual noise steps are processed in the same way as the actual detector signal. (ref: E.S. Goulding, Nucl. Instr. Meth. 100(1972)493]

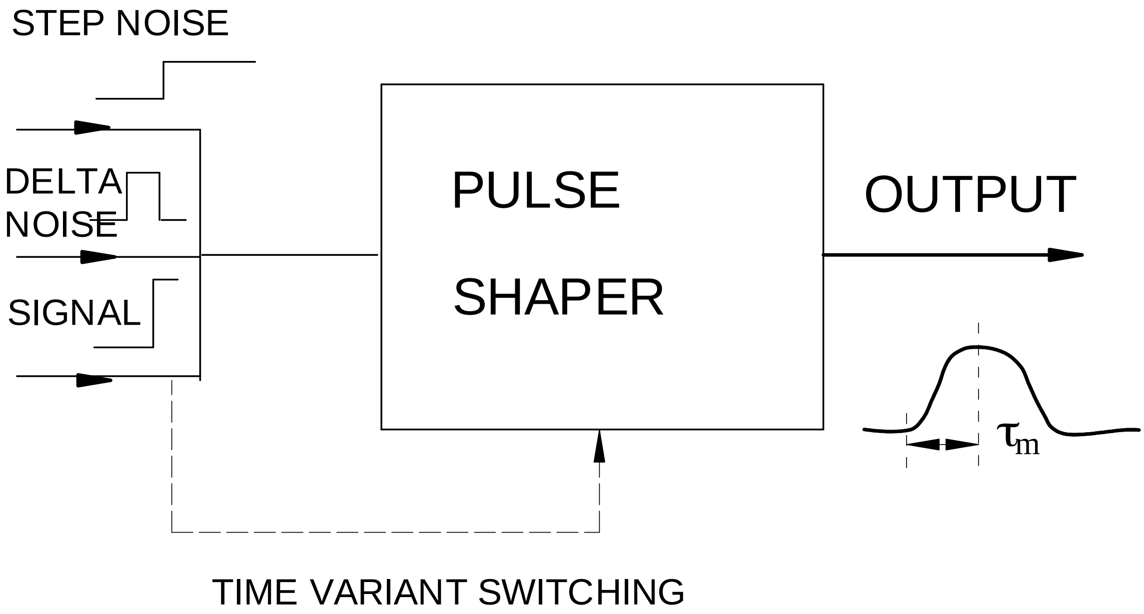


Fig6. Amplifier shaping

Intuitive picture of the effect of the shaping time

A spectroscopic amplifier first differentiates the preamplifier signal to remove the DC component and applies an active or passive integration to generate a pulse of a few micro sec width and a flat top. A qualitative picture of the dependence of the signal and noise with the integration time can be obtained by integrating all the information for a measurement time T. The effective bandwidth Δf ~ 1/T and the frequency domain of interest f ~ 1/T. We have then :

1) Signal out ~ T
2) r.m.s. Delta noise ~ $T^{1/2}$ and
3) r.m.s. step noise ~ $T^{3/2.}$

The above analysis indicates that the signal to noise ratio improves with the shaping time for delta noise and deteriorates for step noise (fig 5).

$$[Noise]^2 \sim \left[A/T + BT \right]$$ where T is the shaping timed A,B depend on the shape of the output pulse. The total noise has a minimum value $\{AB\}^{1/4}$ when the two noise contributions are equal. For semiconductor detectors, this corresponds to a shaping time of 1- 10 μsec depending on the leakage current of the detector. For low-leakage detectors, the step noise contribution is small, and the delta noise is reduced by applying a larger shaping time.

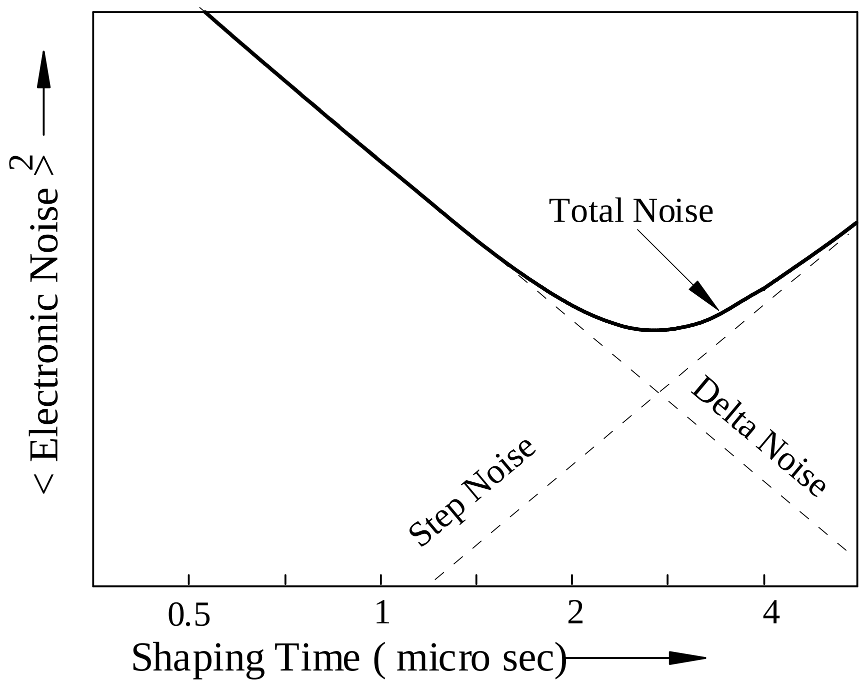


Fig 6. Noise in a typical detector as a function of integration time

Dependence of signal noise on pulse shaping

Let us consider first a passive (time-invariant) shaping based on CR-RC combinations. Such a device would reduce the total bandwidth of the output signal and therefore reduce the white noise contribution. For a step input, the shaper produces an output pulse that rises above noise level, peaks at a time τ_m and then decreases back to noise level. The shape of the output pulse, corresponding to a step input at t=0 is R(t) having a maximum value R_{max} at $t = \tau_m$.

A step function has the property u(t) = 0 for $t < 0$

1 for $t \geq 0$

The normalised pulse shape W(t) = $R(t)/R_{max}$ is also known as the Weighting function.

A step noise of unit magnitude happening at t=0 would produce an output $R(\tau)$ at a time $t = \tau$. Consequence a random sequence of noise step pulses happening at all $t < 0$ hold produce a step noise (adding all the noise components in quadrature)

$$\langle \textbf{STEP NOISE} \rangle^2 \quad \propto \quad \int_0^\infty [R(t)]^2 dt$$

A delta noise can be considered as the time derivative of a step function. Consequently the delta noise is proportional to the time derivative of the output pulse shape :

$$\langle \textbf{DELTA NOISE} \rangle^2 \quad \propto \quad \int_0^\infty [R'(t)]^2 dt$$

The dependence of the signal-to-noise ratio on the pulse shaping is governed by the parameters

$$\text{Step noise index} \qquad \langle N_s^2 \rangle \quad = \quad \int_0^\infty [R(t)]^2 dt \; / \; R_{max}^2$$

and

$$\text{Delta Noise Index} \qquad \langle N_\Delta^2 \rangle \quad = \quad \int_0^\infty [R'(t)]^2 dt \; / \; R_{max}^2$$

The entire effect of the pulse shaper on noise is contained in the $\langle N_s^2 \rangle$ and $\langle N_\Delta^2 \rangle$. It also appears that while the mean square step noise is proportional to the time scale of the pulse shape, the mean square delta noise in inversely proportional to the time scale. We can thus reduce the delta noise contribution by using a longer shaping time but this would result in an increase in step noise.

The product $[\langle N_s^2 \rangle \langle N_\Delta^2 \rangle]^{1/4}$ is dimensionless and is proportional to the minimum r.m.s. noise when the shaping time is optimised. The noise-limiting properties of different types of shaping can be evaluated by comparing the corresponding figures of merit $[\langle N_s^2 \rangle \langle N_\Delta^2 \rangle]^{1/4}$.

The best noise performance is obtained for 'Cusp' shaping $V(x) = e^{-|x|}$ where $x = t/\tau$ is a dimensionless parameter. This shape is however not suitable for pulse processing as the recovery to baseline is very slow. The figures of merit for some typical shaping amplifiers are tabulated below. This shows that Gaussian shaping gives a lower noise figure than CR-RC or CR-(RC)^2 filtering.

Table I : Summary of noise indices for typical systems

Shaping	Analytical form	$\langle N_s^2 \rangle$	$\langle N_\Delta^2 \rangle$	$[\langle N_s^2 \rangle \langle N_\Delta^2 \rangle]^{1/4}$
Cusp	$e^{-\lvert x \rvert}$	1.0	1.0	1.0
CR-RC	$x.e^{-x}$	1.87	1.87	1.37
CR-(RC)^2	$x^2 e^{-x}$	1.71	1.28	1.22
CR-RC-CR	Bipolar			1.88
Gaussian	$\exp(-x^2/2)$			1.12
7 pole	$x^7 e^{7(1-x)}$	0.67	2.53	1.14

approx.				
4 pole approx.	$x^4\ e^{4(1-x)}$	0.90	2.04	1.16
Triangular	$\|1-x\|$	0.67	2.0	1.07
Gated int. with Gaussian prefilter	$T_G = 2.5\ \tau_0$	2.07	1.47	1.32

The total noise contribution, in terms of keV units, is given by :

$$\Delta E_n = 2.35\ \varepsilon/q\ [\ (\ q\ I_d + 2kT/R_{eq}) \langle N_s^2 \rangle + 4\gamma kT\ \ C_{total}{}^2/g_m\ \langle N_\Delta^2 \rangle\ ]$$

where ΔE_n is the FWHM due to the electronic noise in eV, ε is the mean energy required to create one electron-hole pair in the detector and q is the electronic charge.

Selection of components in a preamplifier

Let us look at the criteria for selection of various components used in a preamplifier. The choice is dictated by various conflicting requirements like low noise, high count rate capability, fast response and immunity to detector capacitance variations.

Biasing Resistance

In an ac coupled preamplifier, the biasing resistance is used to provide bias to the detector which is blocked by the coupling capacitance C_s. Since the biasing resistance is a source of step noise, its value is governed by the leakage current of the detector. For detectors of low leakage current (< 1 nA), biasing resistances ~ 1 GΩ can be used (cooled semiconductor detectors and ionisation counters). Resistance values ~ 100 MΩ are used for normal room temperature detectors. For extremely leaky detectors (current > 1 μA), the biasing resistance has to be reduced to 10 MΩ .

The coupling capacitance C_s should have a value large compared to the input capacitance of the preamp with feed back. The leakage current for this high voltage capacitance should be < 1pA so that it does not contribute to noise. Typical value of the coupling capacitance is 10nF with a voltage rating larger than the maximum bias voltage.

For cooled detectors, the biasing resistor and the coupling capacitance can be removed altogether by direct coupling. The detector is insulated from ground and the output charge (and leakage current) flows into the gate of the input FET (fig 7). A low pass RC filter is added in the high voltage side to eliminate the ripple from the HV supply. Care should be taken to apply the preamplifier supply first before applying high voltage.

Feedback Capacitance

The feedback capacitance dictates the gain of the charge sensitive loop. Its value is typically 0.5 - 2 pF . It should be highly stable and contribute negligibly to the input stray capacitance.

To reduce stray capacitance, the feedback capacitance and the coupling capacitance for the test input are generally formed on a PCB on which the gate of the FET is mounted.

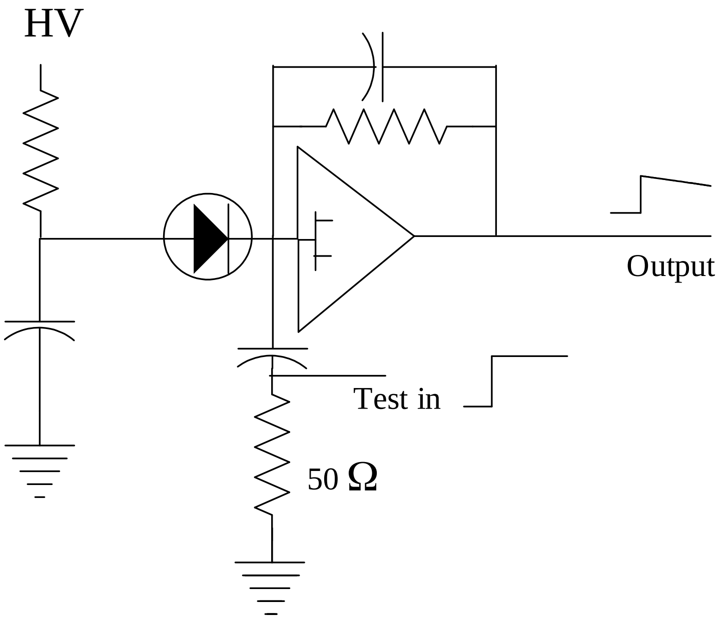


Fig 7. DC coupling of detectors

Feedback Resistance R_f

The feedback resistance has three important functions (i) it allows a slow recovery of the step pulse to zero (ii) provide gate bias and (iii) for direct coupled preamplifiers, the current path for the detector leakage current and/or electron-hole current due to energy deposited in the detector. It also acts as a source of step noise. Typical value of the feedback resistance is 100 MΩ - 1 GΩ.

At high count rates, the current flowing through the feedback resistor would cause a DC shift of the operating point of the FET gate voltage, and limit the maximum deposited energy that can be handled by the preamplifier. For an energy deposition rate of 2.10^5 MeV/sec, the electron-hole current is $(2.10^{11} /3)* 1.6 . 10^{-19} = 10^{-8}$ A producing a voltage drop of 10V across a 1 GΩ feedback resistance. The energy rate can be increased substantially by reducing the feedback resistance with a corresponding increase in noise. The table below shows the increase of noise as the feedback resistance is decreased . The feedback resistance also dictates the decay time C_fR_f of the output wave form. Resistances with very value (> 2 GΩ) have poor high frequency response. As a result, the decay time of the resultant step signal has multiple time constants which would be difficult to compensate using pole-zero technique. Other methods of gate bias control would be preferred for low noise or high charge rate applications.

Table 2 : variation of detector noise with feedback resistance

Resistor Value	122 keV	1332 keV
2.0 GΩ	1.00 keV	
1.0 GΩ		1.81 keV
0.5 GΩ	1.08 keV	1.93 keV
0.2 GΩ	1.25 keV	2.13 keV

The resistive feedback for controlling gate voltage is known as dynamic feedback. For ultra-low noise applications, it is necessary to remove the resistive feedback path. The charge impulses (Q_{in}) from the detector would result in an equivalent charge stored on the feedback capacitance C_f producing a step function $V_o = Q_{in} / C_f$. Subsequent impulses increase the output to a limit at which point the comparator fires an LED directed at the gate of the FET(fig 8). The light pulse momentarily shorts out the FET gate-source junction thus discharging C_f and resetting the preamplifier.

During the discharge of C_f, the preamplifier produces a large negative-going output. An inhibit signal is generated by the processing circuitry to prevent collection of data during the transient reset time. Due to the generation of light-activated surface states in FET, the recovery times are quite large, and pulsed-optical feedback is used only for low count rate applications.

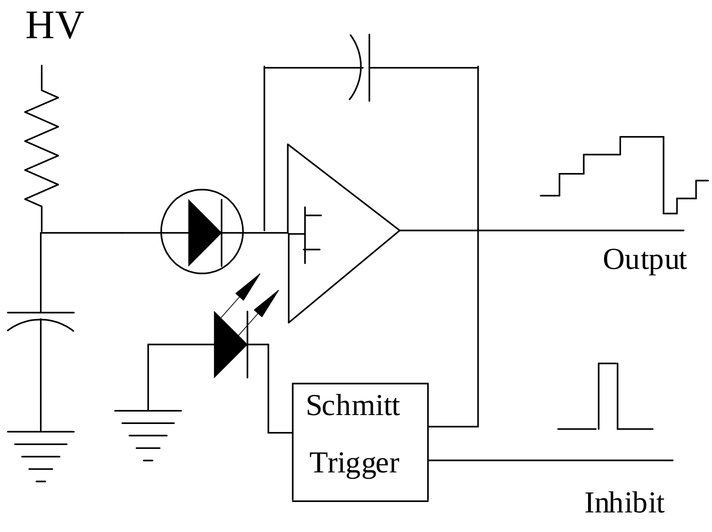


Fig 8: Pulsed Optical Feedback System

For high count-rate applications requiring pulsed feedback, transistor reset preamplifiers are used . The feedback capacitor is discharged by a transistor switch connected to the FET input gate (fig 9} . This adds to some capacitance and noise to the input circuit, but this is acceptable in high count rate applications. Compared to the RC preamplifier where a small value of feedback resistance is used to increase the energy rate of the preamplifier, pulsed transistor rest amplifier would show less noise during the time the transistor is off.

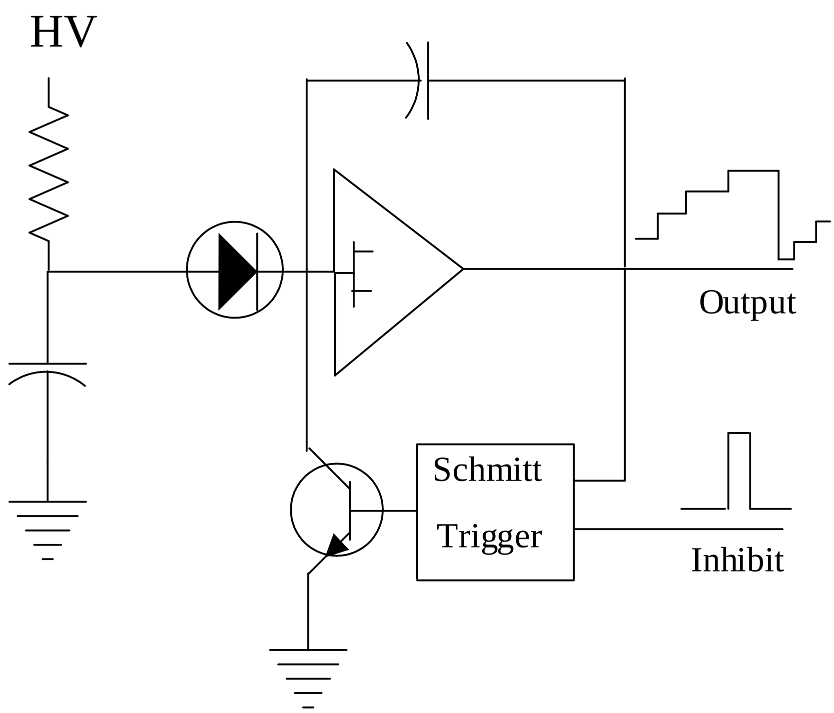


Fig 9. Transistor Reset Preamplifier

Selection of FET

For low noise applications, the choice of FET is critical. It should contribute little to the noise and must have good high frequency response for timing applications. It should have low flicker noise and generation-recombination noise due to the traps present in the gate depletion region of the FET. Compared to BJT and MOSFET devices, JFET devices show lower noise and are almost universally used for the input device in a low noise preamplifier.

The input mean square shot noise is $\propto (C_D + C_{iss})^2/g_m$ where C_D is the detector capacitance and C_{iss} is the input gate capacitance of the FET. For a given family of FETs, the transconductance g_m is proportional to the gate capacitance. (This can be seen by putting two FETs in parallel with their gates connected together.) The quantity $(C_D + C_{iss})^2/C_{iss}$ is minimum when $C_D \approx C_{iss}$. [For semiconductor detectors, the capacitance is inversely proportional to the depletion depth having a value ~ 100 pF μm./mm^2] One should therefore use a selected low noise FET with large transconductance and input capacitance matching the detector capacitance. For cooled detectors, to minimise thermal noise, the FET is also cooled to ~ 140° K where the FET noise is minimum. In addition, the feedback resistance and capacitance are also cooled and kept within the vacuum enclosure of the detector to minimise stray capacitance. For detectors with large capacitance, a number of FETs can be wired parallel to reduce the deterioration of output noise with input capacitance. The table below shows some of the most commonly used FETs for low noise with their characteristics:

FET	g_m (mA/V)	Ciss (pF)
2N5434	90	30
1SK146	40	75
2N4393	20	14
2N4416	6	4
2N6453	5	1

The FET noise is also dependent on drain current.. For the n-channel depletion FET 2N4416, the FET noise is minimum at I_D ~ 8 mA, V_{DS} ~ 10 V. Some preamplifiers have option for adjusting the drain current for the best noise performance.

Operation Amplifier

The operation amplifier following the FET must have an open loop gain > 10^4 resulting in a dynamic input capacitance of > 10^4 pF. The large feedback factor also helps to keep the output linearity very high (< .05% over the dynamic range with a temperature stability of 0.005% /°C)

The charge sensitivity of a preamplifier is determined mainly by the feedback capacitance C_f. [A capacitance of 1 pF corresponds to an output pulse of 50 mV/MeV]. For small signal application, this may be followed by a further gain stage and pulse shaping to give a decay time of ~ 50 μsec for the step output.

The rise time of the output energy pulse (typically 10 - 50 nsec) depends on the gain-bandwidth product of the operational amplifier and the detector capacitance. Use of the energy signal for timing applications would considerably degrade the timing information from small size detectors with short collection times (< 1 ns). The low noise FETs used for the input to the preamplifier are suitable for amplification in VHF/UHF band. The subsequent gain stages have both limited bandwidth and large propagation delay. As a result, the charge sensitive loop lags behind the voltage pulse at the gate of the input FET. The resultant voltage spike at the drain of the fET has a fast rise time. This fast signal can be amplified by a voltage amplifier and produce a separate fast timing output (< 3 ns rise time). For detectors with large capacitance, the rise time is degraded and some tuning of the rise time compensation circuit is required to prevent oscillations at low input capacitance.

Output Gain

The output gain of the charge sensitive loop should be sufficiently large so that the subsequent amplification stage does not significantly degrade the signal to noise ratio. On the other hand, the signal level per pulse should be sufficiently low to prevent saturation due to pileup of output step pulses.

Due to the random spacing of the successive pulses, two or more pulses often overlap in the preamplifier if the product of pulse rate * decay time is not small compared to one. Provided the output voltage is within the linear operating range of the preamplifier, the pileup usually does not cause any problem as the long tail may be easily removed by CR differentiation at the main amplifier.

For large pulse heights, the preamplifier may saturate at even moderate rates giving rise to severe pulse height distortion. The recommended maximum output height per pulse should be restricted to $V_{max}/4$ where V_{max} is the linear operating range of the preamplifier. A preamp of lower gain and faster recovery time is recommended for high energy deposit rate applications. Typical recommended gains for different applications are tabulated below :

Application	maximum Energy (MeV)	gain (mV/MeV)
X-rays	0.1	500
γ-rays	5	100
ΔE detectors	20	50
E Detectors	200	10

The following points should be taken into consideration in selecting and using a preamplifier :

1) Optimise for detector capacitance.

2) For large input signals or very high count rates, make sure that that the specifications of the preamplifier are not exceeded.

3) Always apply bias slowly letting the preamp to recover in between.

4) **NEVER APPLY BIAS WITH THE PREAMPLIFIER POWER OFF.**

NETWORK ANALYSIS

The frequency response of various networks like filters and feedback loops can often be understood in terms of a frequency response F(ω) where ω = 2πf is the angular frequency. The Fourier transform of a function is defined as:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

If $F_{in}(\omega)$ and $F_{out}(\omega)$ are the Fourier transforms of the input and output pulse shape, the the gain of the network is given by

$$G(\omega) = \frac{F_{out}(\omega)}{F_{in}(\omega)}$$

G(ω) is in general complex, so that the network provides both gain and phase shift.

For a circuit containing resistance, inductance and capacitance, the relation between the input and output can be calculated using Kirchhoff's current and voltage Laws using the complex impedances R, $j\omega L$ and $1/j\omega C$ respectively.

For finding the pulse response for a network, the steady state solutions obtained by using Fourier analysis are not adequate, and one has to look for the transient solutions as well. By noting that the voltage across a capacitor is $\int idt / C$ and the back emf across an inductance is $-\int Ldt$ it is possible to rewrite the Kirchhoff equations in terms of integral and differential equations. I would like to show in the following section that the same solutions can be obtained quite easily using Laplace transform without going through the complex integral equations.

The Fourier transform of a non-periodic function does not always exist. For example, consider the step function u(t) :

$$u(t) = \begin{cases} 1 & t>0 \\ 0.5 & t=0 \\ 0 & t<0 \end{cases}$$

This integral does not converge as the value of u(t) does not diminish as t increases. This problem can be resolved by adding an exponentially damping term to the integral.

The Laplace transform is defined as the Fourier transform of $f(t)\, e^{-ct}$:

$$F(c + j\omega) = \int_0^{\infty} f(t)\ e^{-(c + j\omega)t} dt$$

Note that both ω and c have the dimensions of frequency. Defining a complex frequency $s = c + j\,\omega$ the Laplace transform of f(t) can be written as

$$F(s) = \int_0^{\infty} f(t)\ e^{-st} dt$$

The Laplace transform of the step function becomes, by direction integration, $1/s$. This is defined at all values of the complex variable s except at $s=0$. The Laplace transform for commonly found functions are tabulated below.

Table A : Laplace Transform of common functions

Function	*Analytical form*	*Laplace transform*
Step function	$u(t)$	$1/s$
Exponential decay	$e^{-at}\ u(t)$	$1/(s+a)$
Damped oscillation	$e^{-at}\ \sin(\omega\, t)u(t)$	$\omega\ /[(s+a)^2 + \omega^2]$
Delta Function	$\delta(t)$	1

From the definition of Laplace transform, some additive properties can be observed. These can be summarised as :

Table B : Additive properties of Laplace Transforms

Operation	*Analytical form*	*Laplace transform*
Linearity	$\alpha f_1(t) + \beta f_2(t)$	$\alpha F_1(s) + \beta F_2(s)$
scale change	$f(at)$	$(1/a).F(s/a)$
Time shift	$f(t-a)$	$e^{-as}\ F(s)$
Convolution	$\int f_1(\tau)f_2(t-\tau)d\tau$	$F_1(s)\ F_2(s)$
Derivative	$f^{/}(t)$	$sF(s) - f(0)$
Integral	$\int f(\tau)d\tau$	$F(s)/s$
Damping	$e^{-at} f(t)$	$F(s+a)$
Derivative of F	$-tf(t)$	$F^{/}(s)$
Integral of F	$f(t)/t$	$\int_s F(s)\ ds$

It is, in principle, possible to integral the Laplace transform to obtain the original function f(t). However the most common practice is to first decompose the Laplace function to a manageable form and then a table of inverse Laplace transforms to obtain the desired function. This would become clear when we apply the Laplace transform to obtain solutions for some common network problems.

System Analysis

The basic differential equation for a capacitor C passing a current $i(t)$ caused by a terminal voltage $v(t)$ is :

$$i(t) = C\, d[v(t)]/dt$$

Taking the derivative of the Laplace transform, we get

$$I(s) = C[s\, V(s) - v(0)]$$

and $$V(s) = I(s)/QC + v(0)/s$$

The corresponding impedance function is

$$Z(s) = dV(s)/dI(s) = 1/sC$$

For an inductive element,

$$v(i) = L\, d(i(i)/dt$$

$$V(s) = L[sI(s) - i(0)]$$

The corresponding impedance is :

$$Z(s) = V(s)/I(s) = sL$$

We can therefore do a network analysis in complex frequency domain by replacing $j\omega L$ by sL and $j\omega C$ by sC.

Let us consider some problems of interest in shaping amplifiers. For simplicity we consider only those problems where the initial condition is $v(0)=i(0) = 0$ for at inductances and condensers for $t < 0$.

<u>CR Differentiator</u>:

Let us consider a step function input to a CR- Differentiator. The transfer function for the input to output is:

$$G(s) = R/[R+1/(sC)] \quad \text{where } G(s) = V_{out}(s)/V_{in}(s).$$

For a step input $V_{in}(s) = 1/s$ as can be seen in the table A. The output voltage across R has a Laplace transform

$$V_{out}(s) = G(s) \,.\, V_{in}(s) = sRC/(1+sRC) * 1/s = 1/(s+a) \text{ where } a = 1/RC$$

This has a solution (from table A) $V_{out}(t) = e^{-at} = e^{-t/RC}\, u(t)$

Charge sensitive preamp

We now consider the charge sensitive preamplifier with CR feedback. The input current is delta function $\delta(t)$ with unit charge and the output is

$$V_{out}(s) = - Z_{out}(s) * I_{in}(s) = - R*(1/sC)/(R + 1/sC) = -R/(1+sRC)$$

By inverting the transform, we get $v_{out}(t) = -(1/C) * e^{-t/RC} u(t)$

CR-RC integrator

For CR-RC integrator, the transfer functions for the successive stages are:

$G_1(s) = R/(R+1/sC)$
$G_2(s) = (1/sC)/(R + 1/sC)$ and
$V_{in}(s) = 1/s$ for step input.

Consequently $V_{out}(s) = RC/(1+sRC)^2$

To find the corresponding Laplace transform, we note that $V_{out}(s)$ is the derivative of the function -1/(1+sRC). Consequently the output waveform in time domain is

$$V_{out}(t) = (t/RC) * e^{(-t/RC)}$$

LC Circuit

For a step input to an C-(LR) circuit, the transfer function sis:

$G(s) = (R+sL)/(R + sL + 1/sC)$ so that

$v_{out}(s) = (s+R/l)/(s^2 + S R/l + 1/LC)$

The denominator is a quadratic function is s having complex roots at $-\alpha \pm j\omega$ where $\alpha = R/2L$ and $\omega \approx (LC)^{-½}$.

This has a solution of the form

$$v_{out}(t) \approx e^{-\alpha t} cos(\omega t) \quad \text{for } t > 0$$

which corresponds to an exponentially damped oscillation with frequency determined by the LC time constant, with R providing the damping term.

The transfer functions for practical systems using combinations of *R,L,C* can often be expressed as the ratio of polynomials:

$$F(s) = \frac{P(s)}{Q(s)}$$

where *P(s)* and *Q(s)* are polynomials in s of order m and n respectively with real coefficients. The roots of the polynomial *P(s)* are called the **zeroes** of the Transfer function and the roots of the denominator *Q(s)* are called **poles**. For physical solutions, $m \geq n$, as otherwise the solution would be non-vanishing as $t \rightarrow \infty$.

When all the roots of the polynomial *Q(s)* are unequal, *F(s)* can be expressed in the form

$$F(s) = \sum_k \frac{\alpha_k}{(s - a_k)}$$

with a general solution

$$v_{out}(t) = \Sigma_k \ \alpha_k \ exp(a_k t)$$

The coefficients α_k can be calculated by evaluating the quantity $F(s)(s-a_k)$ at $s = a_k$.

Thus each pole on the negative real axis adds an exponentially decaying term to the output. On the other hand, complex poles with -ve real component contribute to an exponentially decaying oscillatory function. The special case of *p* equal roots would have an expansion term $\propto (s-a_k)^{-p}$ which has a solution

$$v_{out}(t) \ \propto \ t^{\,p} \, exp(a_k t)$$

which can be verified by taking a derivative of the Laplace Transform.

A program ***netview*** has been written to numerically find the coefficients C_k when the transfer denominator *Q(s)* has distinct real roots or pairs of complex roots. In case of complex roots $a \pm j\omega$ and coefficient $C \pm jD$, then corresponding real solution is

$$v(t) = 2\, e^{at} \, [C \cos(\omega t) - D \sin(\omega t)\,]$$

The program can also calculate the time dependence of the resultant pulse shape and display it graphically on screen. Two equal roots are simulated by entering values that are separated in real or complex plane by a few percent.

PROBLEMS FOR NETWORK ANALYSIS.

Preamp output into an amplifier

We now consider the more complicated case of preamplifier output fed into a CR-RC shaping network. By replacing the input waveform to :

$V_{in}(s) = 1/(s+b)$ where $1/b = R_f C_f$ is decay time of the charge sensitive loop.

The Laplace transform of the output waveform is :

$V_{out}(s) = sa / [(s+b)(s+a)^2]$

The output waveform can be found in the inverse Laplace transform table[1]:

$$v_{out}(t) = Ate^{-at} + B(e^{-at} - e^{-bt})$$

where $A = a^2/(a-b)$ and $B = ab/(a-b)^2$. The output waveform has an additional slow recovery time with a time constant e^{-bt} (fig 10).

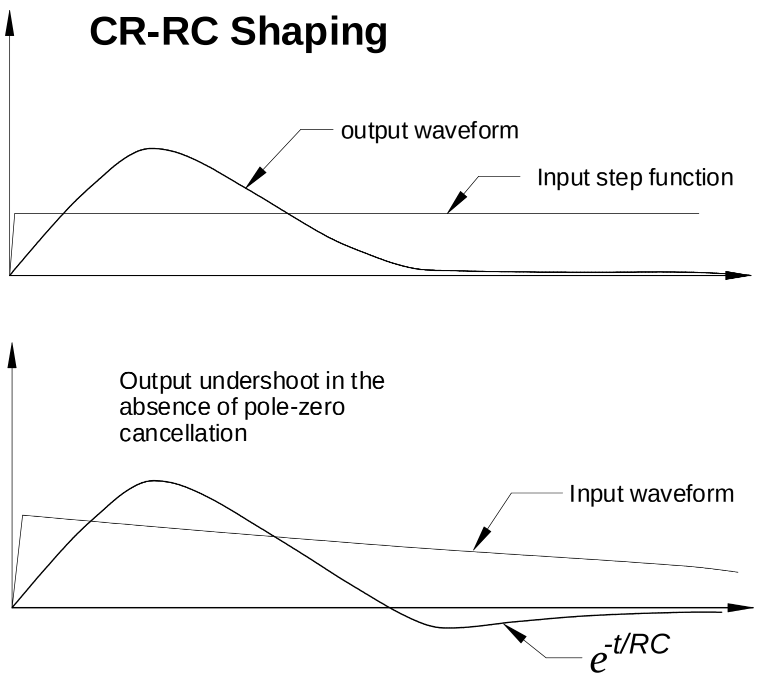


Fig 10. Effect of tail in the input pulse

Let us consider a typical case when $1/a \sim 3$ μs and $1/b \sim 50$ μs. This gives rise to a 10% negative undershoot which decays with a time constant of 50 μsec. At moderate count rates, there is a high probability that the second pulse would be riding on top of the tail of the first pulse. This would result in a loss of resolution as the

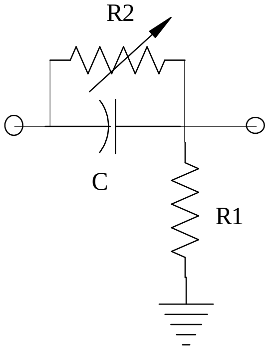
[1] A transfer function of the form (s+d)/[(s+a)(s+b)(s+c)] with three poles at s= -a,-b,-c can be decomposed in the form [$\alpha/(s+a) + \beta/(s+b) + \gamma/(s+c)$] having a solution $v(t) = \alpha e^{-at} + \beta e^{-bt} + \chi e^{-ct}$. The linear dependence on t arises as two of the poles coincide in this case due to equal time constants for the CR-RC network.

height of the second peak with respect to ground is reduced due to the undershoot in the base line.

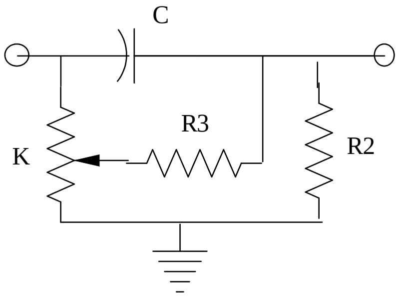
In order to remove the baseline undershoot, the pole *(s+b)* in the denominator of the transfer function should be eliminated. This technique is known as **pole-zero cancellation**, and can be achieved by adding a fraction α of the input pulse to the output of the first differentiator. This modifies the transfer function from $1/(s+a)$ to

$$[1/(s+a) + \alpha] = \alpha(s+a+1/\alpha)/(s+a).$$

If α is chosen to be $\alpha = 1/(b-a)$ then the numerator of the Transfer function has a zero at $s = -b$ which would cancel out the corresponding pole in the Denominator. This method of removing the undershoot of the output waveform is known as **Pole-zero cancellation.** Practical circuits for doing this are shown in fig 11.



$$G(s) = \frac{R1}{R1 + \dfrac{R2 \,.\, 1/Cs}{R2 + 1/Cs}}$$



$$G(s) = \frac{R2}{R2 + 1/Cs} + \frac{KR2}{R2 + R3}$$

Fig 11. Pole-zero network

For long time constants, the second circuit is favoured as it requires a modest value of the circuit components.

Generation of complex poles

The CR/RC networks would normally have only real poles on the negative axis. To generate complex poles, one would require both L & C in the same loop. The need of an inductance can however be avoided using an active feedback loop using operational amplifiers.

Fig 12 shows an active filter amplifier with resistance and capacitances only. The first op amp has infinite gain and the second acts as a buffer with unity gain. The circuit has the advantage that the gain can be varied by changing the coupling resistance R3 which does not have any effect on the poles of the resultant filter.

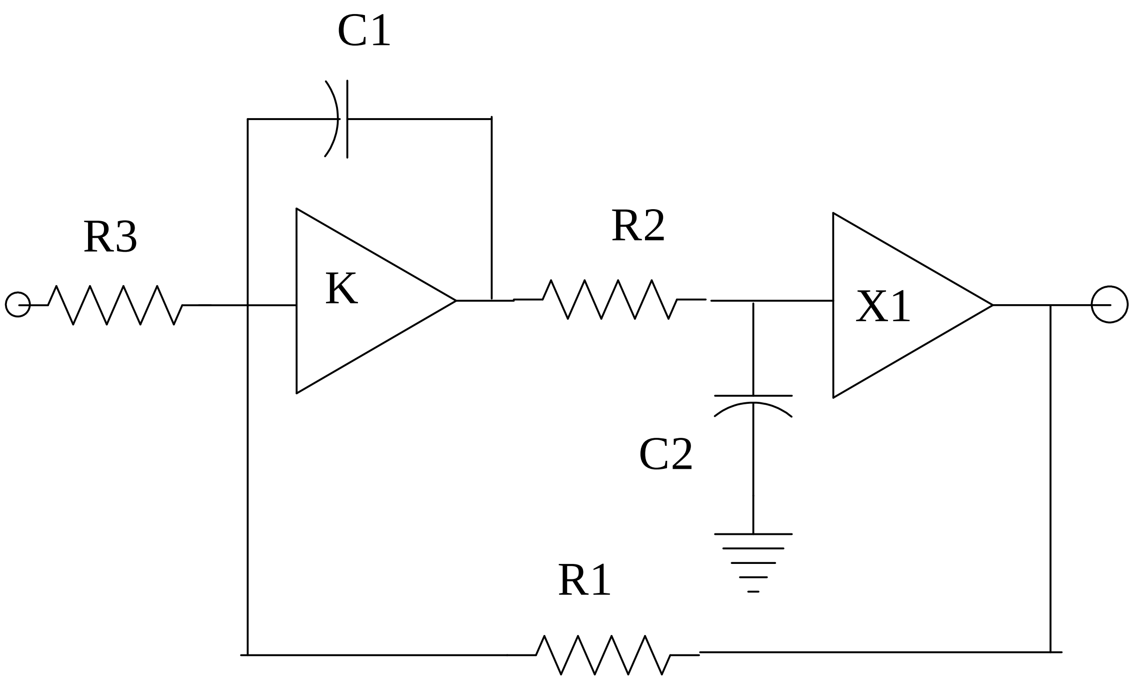


Fig 12: Active filter amplifier unit

The voltage transfer function of the circuit is given as :

$$H(s) = -\left(\frac{R_1}{R_3}\right)\left(\frac{1}{1 + sR_1C_1 + s^2 R_1C_1R_3C_3}\right)$$

This has complex roots if $4R_2C_2 > R_1C_1$

The active filter circuit used in ORTEC amplifiers, shown in fig 13 is known as SALLEN-KEY Filter. For equal values of resistor and capacitance, the network has complex roots.

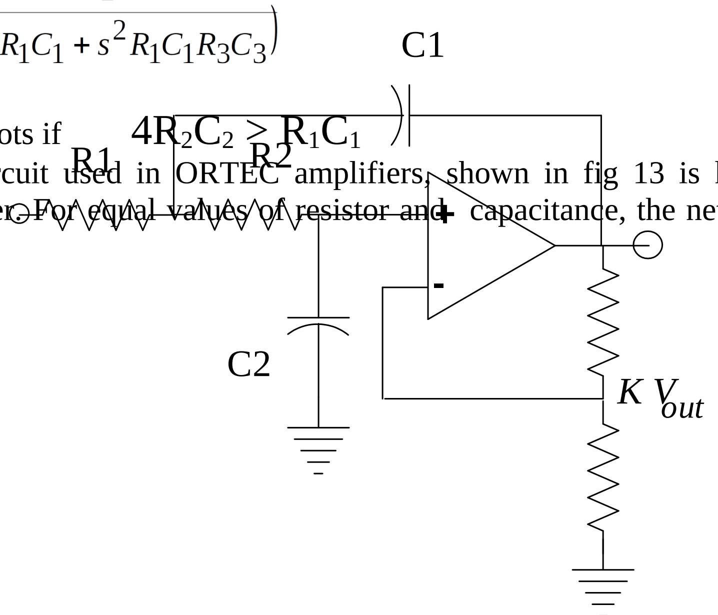


Fig 13: Active filter amplifier unit (ORTEC design)

Synthesis of an output pulse shape

The synthesis of an output waveform is generally done in two steps. First the Fourier transform of the of the wave form is made from which the Transfer function H(s) can be obtained :

$$H(j\omega)H(-j\omega) = |F(\omega)^2|$$

Secondly, $H(s)$ is expanded in the form

$$H(s) = P(s)/Q(s)$$

where *P(s)* and *Q(s)* are polynomials. By more and more terms in the series, one can get better approximations to the final waveform. The real roots of the polynomial *Q(s)* of the form *(s-a)* can be realised with an integrator [with a negative root at $a = -1/RC$]. A pair of self-conjugate complex roots can be realised with active filter circuits mentioned above.

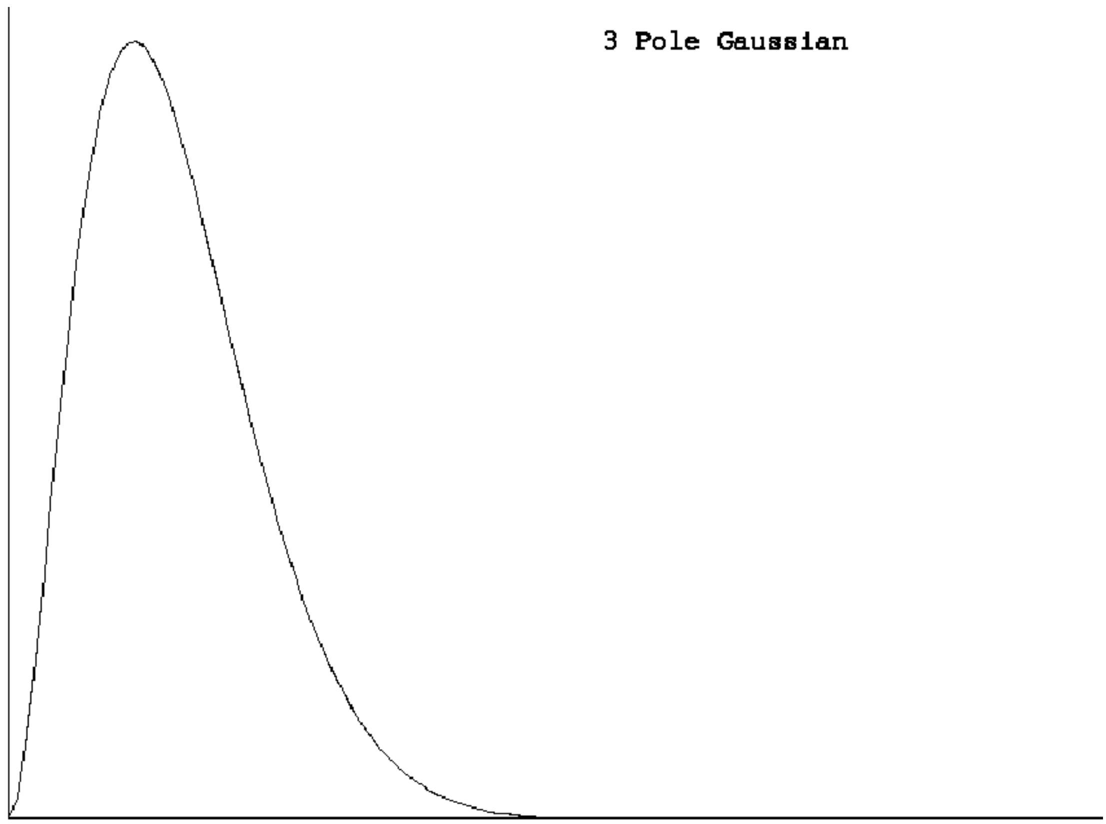
A Gaussian integrator plays a very special role in pulse formation due to its narrow width and good signal-to-noise ratio. One way of achieving it is to use a RC differentiator followed by an infinite series of CR integrators. The corresponding transfer function is

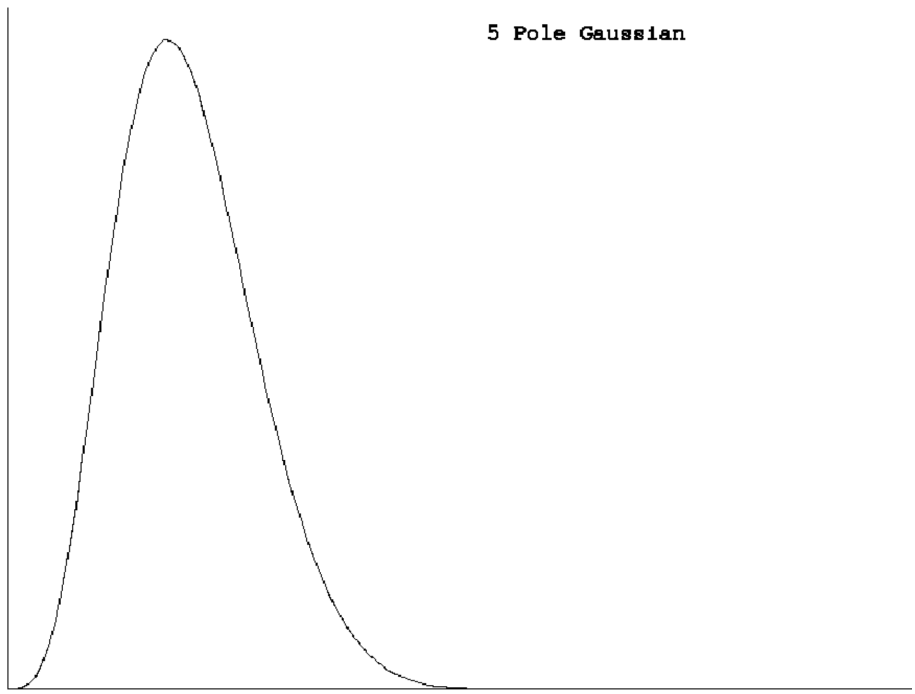
$\frac{s}{1+s}\frac{1}{(1+s)^m}$ as $m \to \infty$. The convergence is however very small requiring a very large number of integrator elements.

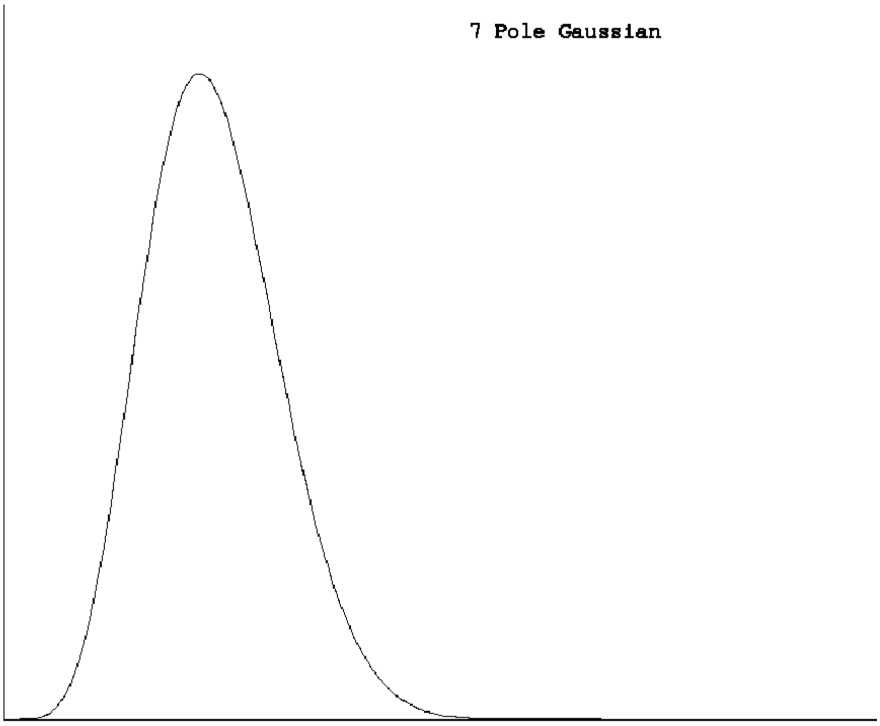
Ohkawa et al [Nucl. Instr. Meth 138(1976)85] have described a truncation scheme where the shaping function corresponds to differentiator followed by a series of active integrators with complex poles. The locations of the poles are tabulated as follows:

Number of poles	Differentiator pole	Active Integrator poles	
		Real part	Imaginary part
3	-1.263	-1.149	±0.786
5	-1.477	-1.417 -1.204	±0.598 ±1.299
7	-1.661	-1.623 -1.495 -1.234	±0.501 ±1.045 ±1.711
4 (Ortec design)	-1.0	-3.0 -1.0	0. ±0.8

An inspection of the table indicates that the imaginary poles are approximate multiples of each other. Thus the undershoot created by the first oscillatory term is compensated by the higher harmonics. The output waveforms, generated by ***netview***, are shown in fig 14. It can be seen that with an increasing number of poles, the output waveform approaches Gaussian form with nearly symmetric shape. For comparison, the active filter unit used by ORTEC is composed of a differentiator, an RC integrator and an active integrator with complex poles.


3 Pole Gaussian


5 Pole Gaussian



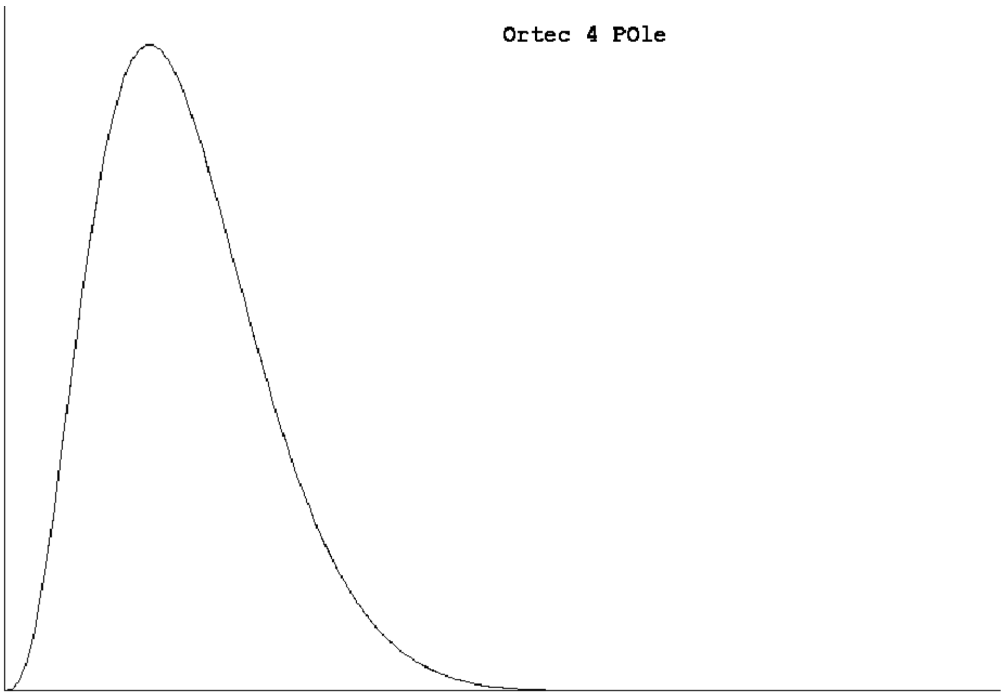


Fig 14. Pseudo-Gaussian Line shapes for filters of various order

Apart from $(CR)^n$ integrator network, there is another class of network known as sine network having the output waveform:

$$F(t) = e^{-3t} (\sin t)^n$$

The waveform is that of a highly damped sine wave raised to the nth power, where n is the number of low pass sections contained in the network. The damping is so large that only the first half cycle is significant, and the next half cycle is lower in amplitude by a factor of ~ 10^{-4}.

A Quasi-Triangular waveform can be generated by adding three sine components :

$$F(t) = e^{-3t}\,[7.64\sin^2 t + 2.53\sin^4 t + 37.74\sin^6 t]$$

This has poles at -3, ($-3\pm 2j$), ($-3\pm 4j$), ($-3\pm 6j$) respectively. The shape is obtained by summing fractions of the outputs from several active integrators that make up the basic network. The waveforms for individual components are shown in fig 13. The slow rise for the first term is compensated by the fast rise from the first term.

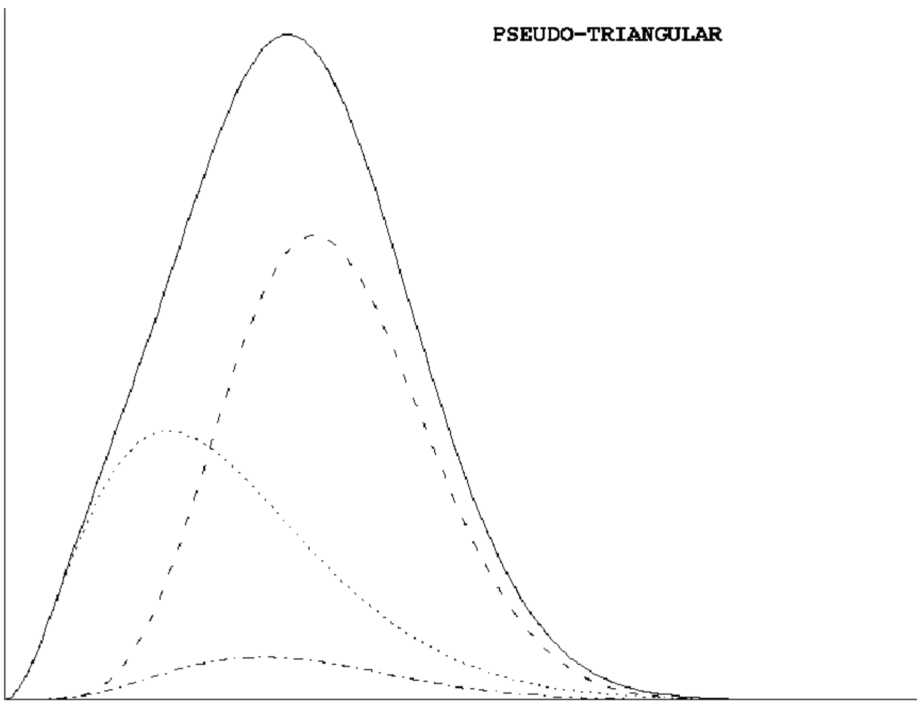


Fig 15 Quasi-Triangular Wave shape with $\sin^2$ (····), $\sin^4$(·-·-·-), $\sin^6$ (- - -) and sum (—)